Algebra of Functions

Objectives

- To define the sum, difference, product, and quotient of functions.
- To form and evaluate composite functions.

Basic function operations

- **_Sum_** $(f+g)(x)=f(x)+g(x)$
- **_Difference_** $(f-g)(x)=f(x)-g(x)$
- **_Product_** $(f \cdot g)(x)=f(x) \cdot g(x)$
- **_Quotient_** $(f/g)(x)=\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)},\ g(x)\neq 0$

Function, domain, & range

- The **domain** of a function is the set of all "first coordinates" of the ordered pairs of a relation.
- The **range** of a function is the set of all "second coordinates" of the ordered pairs of a relation.
- A relation is a function if all values of the domain are unique (they do not repeat).
- A test to see if a relation is a function is the **vertical line test**.
  - If it is possible to draw a vertical line and cross the graph of a relation in more than one point, the relation is not a function.

Example 1

- Find each function and state its domain:

$$f(x)=\sqrt{x+1};\ g(x)=\sqrt{x-1}$$

- $f+g$ $\quad (f+g)(x)=\sqrt{x+1}+\sqrt{x-1};\ D_{f+g}=\{x : x\ge 1\}$
- $f-g$ $\quad (f-g)(x)=\sqrt{x+1}-\sqrt{x-1};\ D_{f-g}=\{x : x\ge 1\}$
- $f\cdot g$ $\quad (f\cdot g)(x)=\left(\sqrt{x+1}\right)\left(\sqrt{x-1}\right)=\sqrt{x^2-1};\ D_{f\cdot g}=\{x : x\ge 1\}$
- f/g $\quad (f/g)(x)=\dfrac{\sqrt{x+1}}{\sqrt{x-1}};\ D_{f/g}=\{x : x>1\}$

Example 2

- The efficiency of an engine with a given heat output, in calories, can be calculated by finding the ratio of two functions of heat input, *D* and *N*, where ***D*(*i*) = *i* – 5700** and ***N*(*i*) = *i*.**
  - Write a function for the efficiency of the engine in terms of heat input (*i*), in calories.

$$E(i) = \frac{i - 5700}{i}$$

  - Find the efficiency when the heat input is 17,200 calories.

$$E(17{,}200) = \frac{17{,}200 - 5700}{17{,}200} \approx 0.67$$

Composition of functions

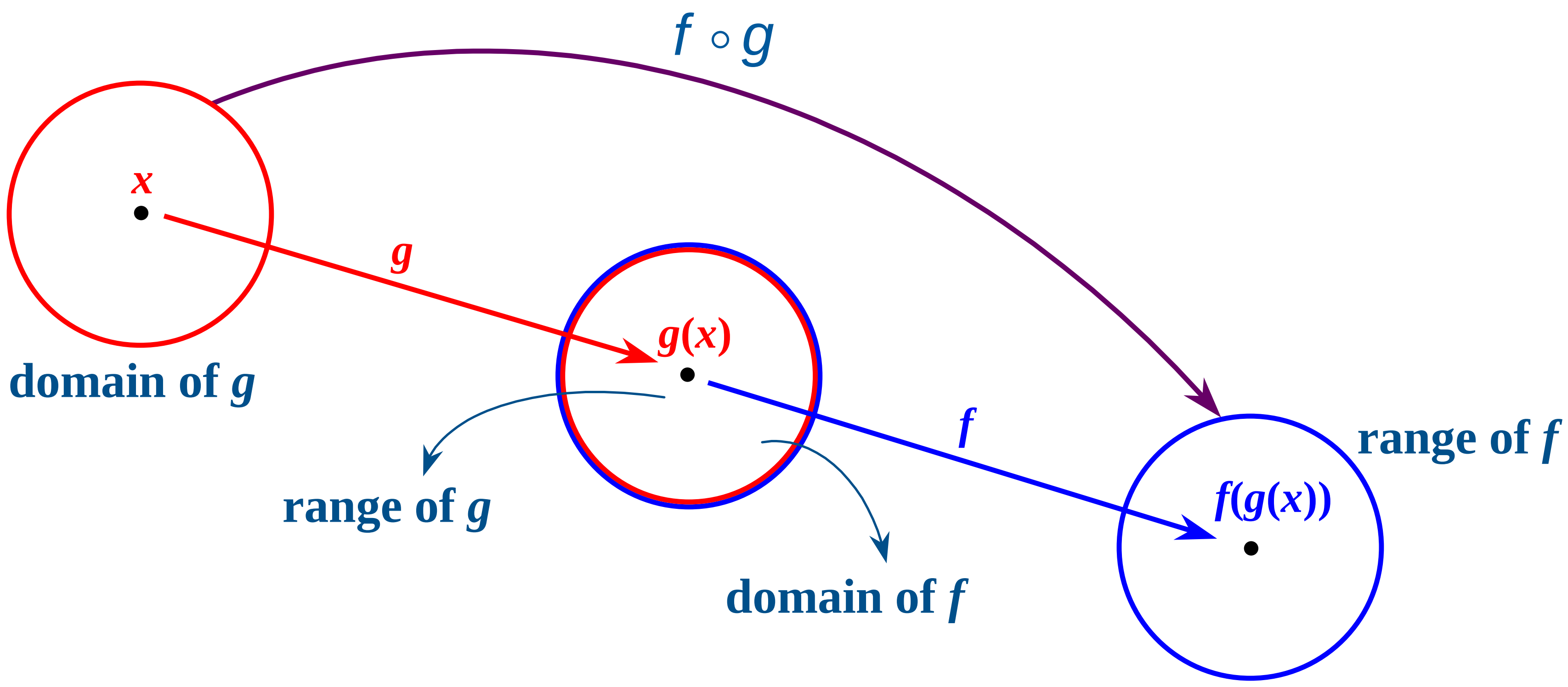
- Composition of functions is the successive application of the functions in a specific order.

- Given two functions *f* and *g*, the **composite function** $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$ and is read "*f* of *g* of *x*."

- The domain of $f \circ g$ is the set of elements *x* in the domain of *g* such that *g*(*x*) is in the domain of *f*.

  - Another way to say that is to say that "the range of function *g* must be in the domain of function *f*."

A composite function



Example 3

$f(x) = x - 3$

$g(x) = 2x^2 - 1$

- Evaluate $(f \circ g)(x)$ and $(g \circ f)(x)$:

$$f(g(x)) = (2x^2 - 1) - 3$$
$$= 2x^2 - 4$$

$$g(f(x)) = 2(x - 3)^2 - 1$$
$$= 2(x^2 - 6x + 9) - 1$$
$$= 2x^2 - 12x + 18 - 1$$

$$(f \circ g)(x) = 2x^2 - 4$$

$$(g \circ f)(x) = 2x^2 - 12x + 17$$

- You can see that function composition is not commutative!

Example 4

- Find the domain of $(f \circ g)(x)$ and $(g \circ f)(x)$:
  - $f(x) = x - 1$
  - $g(x) = \sqrt{x}$

$$(f \circ g)(x) = \sqrt{x} - 1 \qquad D_{f \circ g} = \{x : x \geq 0\}$$

$$(g \circ f)(x) = \sqrt{x - 1} \qquad D_{g \circ f} = \{x : x \geq 1\}$$

(Since a radicand can't be negative in the set of real numbers, the radicand must be greater than or equal to zero. This is what limits the domain.)

Example 5

- The number of bicycle helmets produced in a factory each day is a function of the number of hours (*t*) the assembly line is in operation that day and is given by $n = P(t) = 75t - 2t^2$.
- The cost *C* of producing the helmets is a function of the number of helmets produced and is given by $C(n) = 7n + 1000$.
  - Determine a function that gives the cost of producing the helmets in terms of the number of hours the assembly line is functioning on a given day.
  - Find the cost of the bicycle helmets produced on a day when the assembly line was functioning 12 hours. *(solution on next slide)*

$C(n) = 7n + 1000$ $\quad n = P(t) = 75t - 2t^2$

Solution to Example 5:

- Determine a function that gives the cost of producing the helmets in terms of the number of hours the assembly line is functioning on a given day.
- $$\begin{aligned} \text{Cost} = C(n) &= C(P(t)) \\ &= C(75t - 2t^2) \\ &= 7(75t - 2t^2) + 1000 \\ &= -14t^2 + 525t + 1000 \end{aligned}$$

  - Find the cost of the bicycle helmets produced on a day when the assembly line was functioning 12 hours.

$$C = -14t^2 + 525t + 1000 = \$5284$$

Review

- If $f(x) = 2x + 1$ and $g(x) = x^2$, find $f(g(x))$.
  - Find $g(f(x))$.
  - What is the domain of $g(f(x))$?
- Consider the functions $a(x) = \sqrt{x+1} \cdot \sqrt{x-1}$ and $b(x) = \dfrac{\sqrt{x+1}}{\sqrt{x-1}}$. Why are their domains different?

Answers to review:

$$f\left(g\left(x\right)\right)=f\left(x^2\right)=2\left(x^2\right)+1=\boxed{2x^2+1}$$

$$g\left(f\left(x\right)\right)=g\left(2x+1\right)=\left(2x+1\right)^2=\boxed{4x^2+4x+1}$$

Domain of $g\left(f\left(x\right)\right)$ is $\left\{x : x \in \Re\right\}$

The domains of the two functions are different because the denominator of *b*(x) cannot be zero.

Summary…

- Function arithmetic – add the functions (subtract, etc)
  - Addition
  - Subtraction
  - Multiplication
  - Division
- Function composition
  - Perform function in innermost parentheses first
  - Domain of “main” function must include range of “inner” function